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A Remark on Compact Automorphism Groups of C^*-Algebras

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A C^*-system over a compact group whose fixed-point algebra has trivial centre and a relative commutant coinciding with the centre of the full algebra is obtained canonically by inducing up from a C^*-system over a closed subgroup with the same fixed-point algebra but which now has trivial relative commutant. © 1986 Academic Press, Inc.

Let $\mathscr{B}$ be a C^*-algebra with unit, G a compact group, and α a strongly continuous action of G by automorphisms of $\mathscr{B}$. We shall denote by $\mathscr{A}$ the α_G-fixed-point subalgebra of $\mathscr{B}$ and by $\mathscr{C}$ the centre of $\mathscr{B}$,

$$\mathscr{A} \equiv \mathscr{B}^\alpha,$$

$$\mathscr{C} \equiv \mathscr{B}' \cap \mathscr{B}.$$

In this article we show that if the relative commutant of $\mathscr{A}$ in $\mathscr{B}$ is $\mathscr{C}$ and $\mathscr{A}$ has trivial centre,

$$\mathscr{A}' \cap \mathscr{B} = \mathscr{C}, \tag{1}$$

$$\mathscr{A}' \cap \mathscr{A} = \mathbb{C}I, \tag{2}$$

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then there is a C^*-algebra $\mathscr{F}$ and an action τ of a closed subgroup H of G such that $\mathscr{A}$ is the τ_H-fixed points of $\mathscr{F}$, the relative commutant of $\mathscr{A}$ in $\mathscr{F}$ reduces to the complex numbers and $\{\mathscr{B}, \alpha\}$ is obtained from $\{\mathscr{F}, \tau\}$ by inducing up from the subgroup H to G. This simple construction is used in an essential way to determine the compact Lie group associated with certain endomorphisms of C^*-algebras [1].

If H is a closed subgroup of the compact group G and τ is a strongly continuous action of H by automorphisms of a C^*-algebra $\mathscr{F}$, the C^*-system $\{\mathscr{C}_\tau(G, \mathscr{F}), \hat{\tau}\}$ obtained from $\{\mathscr{F}, \tau\}$ by inducing up to G is defined as follows. $\mathscr{C}_\tau(G, \mathscr{F})$ is the C^*-algebra of continuous functions X from G to $\mathscr{F}$ (with pointwise *-algebraic operations and supremum norm) satisfying

$$X(hg) = \tau_h(X(g)), \qquad h \in H, \quad g \in G; \tag{3}$$

$\hat{\tau}$ is the action of G on $\mathscr{C}_\tau(G, \mathscr{F})$ defined by $(\hat{\tau}_g(X))(g_0) = X(g_0 g)$; $g, g_0 \in G$, $X \in \mathscr{C}_\tau(G, \mathscr{F})$. (4)

1. Theorem. *Under assumptions* (1) *and* (2) *on the C^*-system $\{\mathscr{B}, \alpha\}$ over the compact group G, there exists a C^*-algebra $\mathscr{F}$, a closed subgroup H of G, a strongly continuous action τ of H on $\mathscr{F}$ and a monomorphism π of $\mathscr{A}$ into $\mathscr{F}$ such that*

(a) $\pi(\mathscr{A}) = \mathscr{F}^\tau$,

(b) $\pi(\mathscr{A})' \cap \mathscr{F} = \mathbb{C}I$,

(c) *$\{\mathscr{B}, \alpha\}$ is isomorphic to $\{\mathscr{C}_\tau(G, \mathscr{F}), \hat{\tau}\}$.*

Moreover, H is unique up to conjugation in G and $\{\mathscr{F}, \tau\}$ is then unique up to isomorphism.

The proof will require some elementary lemmas. First, note that (1) and (2) imply $\mathscr{C}^\alpha = \mathbb{C}I$ so that $g \in G \to \alpha_g | \mathscr{C}$ is an ergodic action of a compact group on a commutative C^*-algebra. The transposed action $g \to \phi \circ \alpha_g^{-1}$ is then an ergodic action by homeomorphisms of the compact Hausdorff space $\sigma(\mathscr{C})$, the spectrum of $\mathscr{C}$. As the action is continuous and G is compact, each point in $\sigma(\mathscr{C})$ has a closed orbit so that ergodicity implies that G acts transitively on $\sigma(\mathscr{C})$. If

$$H = \{g \in G \mid \phi \circ \alpha_g = \phi\}$$

is the stabilizer of some chosen $\phi \in \sigma(\mathscr{C})$, H is a closed subgroup of G and $\sigma(\mathscr{C})$ is homeomorphic to the coset space $H \backslash G$.

Let J_ϕ denote the smallest closed two sided (or equivalently left or right) ideal containing $\ker \phi$, J_ϕ is the closed linear span of

$$\mathscr{B} \ker \phi \mathscr{B} = \mathscr{B} \ker \phi = \ker \phi \mathscr{B}.$$

2. LEMMA. *The ideal J_ϕ is the intersection of the kernels of the* G.N.S. *representations π_ω, where ω ranges over the pure state extensions of ϕ to $\mathscr{B}$. Moreover*

$$\bigcap_{\phi \in \sigma(\mathscr{C})} J_\phi = \{0\}. \tag{5}$$

Proof. Let $\hat{J}_\phi$ denote the intersection described in the statement of the lemma. A pure state ω of $\mathscr{B}$ restricts to a pure state of $\mathscr{C}$, $\omega | \mathscr{C} \in \sigma(\mathscr{C})$, and $\pi_\omega(C) = \omega(C) I$, $C \in \mathscr{C}$. Hence $\ker \pi_\omega \supset \ker(\omega | \mathscr{C})$ so $J_\phi \subset \hat{J}_\phi$. Conversely, every irreducible representation $\tilde{\pi}$ of $\mathscr{B}/J_\phi$ is of the form $\tilde{\pi}(B + J_\phi) = \pi_\omega(B)$, where ω is a pure state of $\mathscr{B}$ and $\omega | \mathscr{C} = \phi$. Since $\mathscr{B}/J_\phi$ is separated by its irreducible representations, $\hat{J}_\phi \subset J_\phi$ and equality is proved. We now have

$$\bigcap_{\phi \in \sigma(\mathscr{C})} J_\phi = \bigcap \{\ker \pi_\omega | \omega \text{ is a pure state of } \mathscr{B}\} = \{0\}.$$

If α is an automorphism of $\mathscr{B}$ and $\phi \in \sigma(\mathscr{C})$, then $\alpha(\ker \phi) = \ker \phi \circ \alpha^{-1}$ so that

$$\alpha(J_\phi) = J_{\phi \circ \alpha^{-1}}.$$

In particular, $\alpha_g(J_\phi) = J_{\phi \circ \alpha_g^{-1}}$ so that the stabilizer H of ϕ leaves J_ϕ globally invariant.

Fixing $\phi \in \sigma(C)$, we write

$$J_g = \alpha_g(J_\phi) = J_{\phi \circ \alpha_g^{-1}}, \qquad g \in G, \tag{6}$$

for brevity so that, in particular, $J_e = J_\phi$, where e is the unit of G. Let $\eta \colon \mathscr{B} \to \mathscr{B}/J_e$ denote the quotient map, $\eta(B) = B + J_e$, $B \in \mathscr{B}$ and let

$$\eta_g = \eta \circ \alpha_g, \qquad g \in G, \tag{7}$$

then Eqs. (5) and (6) give

$$\begin{gathered} \ker \eta_g = \alpha_g^{-1}(J_e) = J_{g^{-1}}, \\ \bigcap_{g \in G} \ker \eta_g = \{0\}. \end{gathered} \tag{8}$$

The C^*-system $\{\mathscr{F}, \tau\}$ is now defined by

$$\begin{gathered} \mathscr{F} = \eta(\mathscr{B}), \\ \tau_h \circ \eta = \eta \circ \alpha_h, \qquad h \in H, \end{gathered} \tag{9}$$

and the induced C^*-system $\{\mathscr{C}_\tau(G, \mathscr{F}), \hat{\tau}\}$, introduced above, will be denoted simply by $\{\hat{\mathscr{F}}, \hat{\tau}\}$. Define a homomorphism

$$B \in \mathscr{B} \to \hat{B} \in \hat{\mathscr{F}} \tag{10}$$

by

$$\hat{B}(g) = \eta_g(B); \qquad B \in \mathscr{B}, \quad g \in G.$$

The map (10) is well defined since, for $h \in H$, $g \in G$, we have by (9),

$$\hat{B}(hg) = \eta_{hg}(B) = \eta \circ \alpha_{hg}(B) = \tau_h \circ \eta \circ \alpha_g(B) = \tau_h(\hat{B}(g)).$$

The map (10) intertwines α and $\hat{\tau}$ since, for g, $g_0 \in G$

$$(\hat{\tau}_g(\hat{B}))(g_0) = \hat{B}(g_0 g) = \eta_{g_0 g}(B) = \eta \circ \alpha_{g_0 g}(B) = \widehat{\alpha_g(B)}(g_0).$$

3. Lemma. *The map $B \in \mathscr{B} \to \hat{\mathscr{B}} \in \hat{\mathscr{F}}$ defines an isomorphism of $\{\mathscr{B}, \alpha\}$ onto $\{\hat{\mathscr{F}}, \hat{\tau}\}$.*

Proof. By (8) and (10), $\hat{B} = 0$ implies $B = 0$ so we have only to establish that (10) has dense image. Given $X \in \hat{\mathscr{F}}$, $\varepsilon > 0$, and $g \in G$ there is a $B \in \mathscr{B}$ with

$$\|X(g') - \hat{B}(g')\| < \varepsilon, \qquad g' \in \Omega,$$

where Ω is an H-stable neighbourhood of g, i.e., $h\Omega = \Omega$, $h \in H$. Since G is compact we can find a finite covering $\Omega_1, \Omega_2, ..., \Omega_n$ of G by H-stable open sets and $B_1, B_2, ..., B_n \in \mathscr{B}$ with

$$\|X(g) - \hat{B}_i(g)\| < \varepsilon, \qquad g \in \Omega_i, \quad i = 1, 2, ..., n.$$

Pick $C_1, C_2, ..., C_n \in \mathscr{C}$ so that $\hat{C}_1, \hat{C}_2, ..., \hat{C}_n$ is a partition of unity by continuous functions constant on left H-cosets subordinate to the covering and let $B = \sum_i C_i B_i$. Then

$$\begin{aligned} \|X(g) - \hat{B}(g)\| &= \Big\| \sum_i \hat{C}_i(g)(X(g) - \hat{B}_i(g)) \Big\| \\ &\leqslant \sum_i \hat{C}_i(g) \, \|X(g) - \hat{B}_i(g)\| < \varepsilon, \qquad g \in G, \end{aligned}$$

completing the proof.

Our original argument at this point identified $\hat{\mathscr{F}}$ with the C^*-algebra defined by a continuous field of C^*-algebras over $H\backslash G$ and appealed to the Stone–Weierstrass theorem for continuous fields [2, Corollary 11.5.3]. G.

K. Pedersen kindly drew our attention to a related result [3, Theorem 2.6] which led us to simplify the proof of this lemma.

Proof of the Theorem. Let $\pi: \mathscr{A} \to \mathscr{F}$ be the restriction of η to the fixed-point algebra $\mathscr{A}$. By (7)

$$\eta_g(A) = \pi(A), \qquad g \in G, \quad A \in \mathscr{A}, \tag{11}$$

so that π is injective by (8). The C^*-systems $\{\mathscr{B}, \alpha\}$ and $\{\hat{\mathscr{F}}, \hat{\tau}\}$ are isomorphic by Lemma 3 so

$$\hat{\mathscr{A}} = \hat{\mathscr{F}}^{\hat{\tau}}. \tag{12}$$

Comparing (10) and (11), $\hat{\mathscr{A}}$ is the set of constant $\pi(\mathscr{A})$-valued functions on G. On the other hand, $X \in \hat{\mathscr{F}}^{\hat{\tau}}$ if and only if $X(g_0 g) = X(g_0)$, $g, g_0 \in G$, i.e., $X(g) = X(e)$, $g \in G$ and $\tau_h(X(e)) = X(h) = X(e)$, $h \in H$. In other words, $\hat{\mathscr{F}}^{\hat{\tau}}$ is the set of constant $\mathscr{F}^{\tau}$-valued functions on G. So (12) can be read as

$$\pi(\mathscr{A}) = \mathscr{F}^{\tau}.$$

Lemma 3 also gives us the equality

$$(\mathscr{A}' \cap \mathscr{B})\hat{} = (\hat{\mathscr{F}}^{\hat{\tau}})' \cap \hat{\mathscr{F}}. \tag{13}$$

Now the restriction of the map $\hat{}$ to $\mathscr{A}' \cap \mathscr{B}$ is essentially the Gelfand transform: the l.h.s. of (13) is the set of all continuous $\mathbb{C}I$-valued functions on G constant on left H-cosets. The r.h.s. is the set of all $\pi(\mathscr{A})' \cap \mathscr{F}$-valued functions in $\hat{\mathscr{F}}$. As the set of values taken by such functions at $e \in G$ is $\pi(\mathscr{A})' \cap \mathscr{F}$, we get

$$\pi(\mathscr{A})' \cap \mathscr{F} = \mathbb{C}I.$$

Thus we have proved (a) and (b) of the Theorem. Part (c) was already proved as Lemma 3 so we are left with the assertions on uniqueness. Let $\{\mathscr{F}_1, H_1, \tau_1, \pi_1\}$ be another choice satisfying (a), (b), and (c) and let $\rho: \mathscr{B} \to \mathscr{C}_{\tau_1}(G, \mathscr{F}_1)$ be an isomorphism realizing (c). The image of the centre $\mathscr{C}$ of $\mathscr{B}$ under ρ is the centre of $\mathscr{C}_{\tau_1}(G, \mathscr{F}_1)$, i.e., the set of continuous $\mathbb{C}I$-valued functions on G constant on left H_1-cosets. Hence there is a $g \in G$ such that

$$\phi(A)\, I = \rho(A)(g), \qquad A \in \mathscr{C}, \tag{14}$$

where $\phi \in \sigma(\mathscr{C})$ is the reference point used to define η. Consequently,

$$H_1 = gHg^{-1}.$$

Modulo an automorphism in α_G, we may now assume that $g = e$ in (14). The homomorphism

$$X \in \mathscr{C}_{\tau_1}(G, \mathscr{F}_1) \to X(e) \in \mathscr{F}_1$$

with kernel J_1, say, gives us an exact sequence

$$0 \to J_1 \to \mathscr{C}_{\tau_1}(G, \mathscr{F}_1) \to \mathscr{F}_1 \to 0.$$

By definition, we have an exact sequence

$$0 \to J_\phi \to \mathscr{B} \to \mathscr{F} \to 0$$

and, since $g = e$ in (14), $\rho(J_\phi) = J_1$. Hence the diagram

$$\begin{tikzcd} 0 \arrow[r] & J_1 \arrow[r] & \mathscr{C}_{\tau_1}(G, \mathscr{F}_1) \arrow[r] & \mathscr{F}_1 \arrow[r] & 0 \\ 0 \arrow[r] & J_\phi \arrow[r] \arrow[u, "\rho"'] & \mathscr{B} \arrow[r] \arrow[u, "\rho"'] & \mathscr{F} \arrow[r] \arrow[u, "\rho_1"'] & 0 \end{tikzcd}$$

defines an isomorphism ρ_1 of $\mathscr{F}$ onto $\mathscr{F}_1$. Since ρ intertwines the actions α and $\hat{\tau}_1$ of G and J_ϕ and J_1 are stable under the restriction of these actions to H, ρ_1 must intertwine the actions τ and τ_1 of H.

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